## Appendix H
# Characteristic Matrix in a Stack of Airy Layers

We model an actual profile for the index of refraction $n(x)$ by a stack of Airy layers (see Sections 4.5 and 4.7). We consider a stack of Cartesian stratified layers where $n'$ is a constant within any given layer but is discontinuous across the boundary between layers. Within a given layer, $n'_A$ is held constant, but across a boundary between two layers, both $n_A$ and $n'_A$ are adjusted to ensure that $\hat{y}$ remains invariant and to best match the profile of the actual index of refraction.

The elements of the characteristic matrix $\boldsymbol{M}[\hat{y}_{A\,j}, \hat{y}_{A\,j-1}]$ for a transverse electric (TE) wave within the $j$th Airy layer are given in Eq. (4.5-6). We introduce a shorthand notation here to reduce the size of the equations:

$$
\left.
\begin{aligned}
F_{j,j-1} &= F\left(\hat{y}_{Aj}, \hat{y}_{Aj-1}\right) = \pi \begin{vmatrix} \mathrm{Ai}\left[\hat{y}_{Aj}\right] & \mathrm{Bi}\left[\hat{y}_{Aj}\right] \\ \mathrm{Ai}'\left[\hat{y}_{Aj-1}\right] & \mathrm{Bi}'\left[\hat{y}_{Aj-1}\right] \end{vmatrix} \\
f_{j,j-1} &= f\left(\hat{y}_{Aj}, \hat{y}_{Aj-1}\right) = \pi \frac{i}{\gamma_{Aj}} \begin{vmatrix} \mathrm{Ai}\left[\hat{y}_{Aj}\right] & \mathrm{Bi}\left[\hat{y}_{Aj}\right] \\ \mathrm{Ai}\left[\hat{y}_{Aj-1}\right] & \mathrm{Bi}\left[\hat{y}_{Aj-1}\right] \end{vmatrix} \\
G_{j,j-1} &= G\left(\hat{y}_{Aj}, \hat{y}_{Aj-1}\right) = \pi i \gamma_{Aj} \begin{vmatrix} \mathrm{Ai}'\left[\hat{y}_{Aj}\right] & \mathrm{Bi}'\left[\hat{y}_{Aj}\right] \\ \mathrm{Ai}'\left[\hat{y}_{Aj-1}\right] & \mathrm{Bi}'\left[\hat{y}_{Aj-1}\right] \end{vmatrix} \\
g_{j,j-1} &= g\left(\hat{y}_{Aj}, \hat{y}_{Aj-1}\right) = -\pi \begin{vmatrix} \mathrm{Ai}'\left[\hat{y}_{Aj}\right] & \mathrm{Bi}'\left[\hat{y}_{Aj}\right] \\ \mathrm{Ai}\left[\hat{y}_{Aj-1}\right] & \mathrm{Bi}\left[\hat{y}_{Aj-1}\right] \end{vmatrix}
\end{aligned}
\right\} \tag{H-1}
$$

Here $\gamma_{Aj}$ is given by

$$
\gamma_{Aj} = \left(2k^{-1} n_j n'_{Aj}\right)^{1/3} \tag{H-2}
$$

It is a constant within the $j$th layer. Here $n_j$ is the index of refraction in the actual medium at the bottom of the $j$th Airy layer, i.e., $n_j = n\left(x_{j-1}\right)$, and at this point, $n_A = n_j$. As we progress through the layer, $n_A(x)$, which is a linear function of $x$, will deviate from $n(x)$ because $n'_A$ differs from $n'\left(x_{j-1}\right)$ and because the curvature and higher-order terms in $n(x)$ produce an additional divergence. Across the upper boundary between the $j$th and $j+$ 1st layers, both $n$ and $n'$ are jointly adjusted to keep $\hat{y}_A$ invariant but to readjust $n_A\big|_{x_j}$ so that $n_A\big|_{x_j} = n\left(x_j\right) = n_{Aj+1}$.

Following Eq. (4.4-4), we define a reference characteristic matrix by

$$
\overline{\boldsymbol{M}}_{j,j-1} = \begin{bmatrix} F_{j,j-1} & \bar{f}_{j,j-1} \\ \overline{G}_{j,j-1} & g_{j,j-1} \end{bmatrix} \tag{H-3}
$$

where

$$\left.\begin{aligned}\tilde{f}_{j,j-1} &= f\left(\hat{y}_{Aj},\hat{y}_{Aj-1}\right) = \pi\frac{i}{\bar{\gamma}_A}\begin{vmatrix}\mathrm{Ai}\left[\hat{y}_{Aj}\right] & \mathrm{Bi}\left[\hat{y}_{Aj}\right]\\ \mathrm{Ai}\left[\hat{y}_{Aj-1}\right] & \mathrm{Bi}\left[\hat{y}_{Aj-1}\right]\end{vmatrix}\\ \tilde{G}_{j,j-1} &= G\left(\hat{y}_{Aj},\hat{y}_{Aj-1}\right) = \pi i\bar{\gamma}_A\begin{vmatrix}\mathrm{Ai}'\left[\hat{y}_{Aj}\right] & \mathrm{Bi}'\left[\hat{y}_{Aj}\right]\\ \mathrm{Ai}'\left[\hat{y}_{Aj-1}\right] & \mathrm{Bi}'\left[\hat{y}_{Aj-1}\right]\end{vmatrix}\end{aligned}\right\} \tag{H-4}$$

Here $\bar{\gamma}_A$ is a constant throughout all Airy layers. Its value is to be set later; however, following the discussion in Section 3.5, we would expect it to be some average value among the layers spanned by the characteristic matrix.

It follows that the difference between the actual characteristic matrix for the $j$th Airy layer and the reference matrix is given by

$$\delta\boldsymbol{M}_{j,j-1} = \boldsymbol{M}_{j,j-1} - \overline{\boldsymbol{M}}_{j,j-1} \doteq \begin{bmatrix}0 & -\bar{f}_{j,j-1}\\ \overline{G}_{j,j-1} & 0\end{bmatrix}\frac{\delta\gamma_{Aj}}{\bar{\gamma}_A} \tag{H-5}$$

where $\delta\gamma_{Aj} = \gamma_{Aj} - \bar{\gamma}_A$, a small but non-zero quantity in general. For thin layers, one can show by using the Wronskian for the Airy functions that $\delta\boldsymbol{M}_{j,j-1}$ can be written as

$$\delta\boldsymbol{M}_{j,j-1} \to \delta\boldsymbol{M}_j \doteq \begin{bmatrix}0 & i\bar{\gamma}_A^{-1}\\ i\bar{\gamma}_A\hat{y}_{Aj} & 0\end{bmatrix}\frac{\delta\gamma_{Aj}}{\bar{\gamma}_A}d\hat{y}_{Aj} \tag{H-6}$$

Here $d\hat{y}_{Aj} = \hat{y}_{Aj} - \hat{y}_{Aj-1}$, which will approach zero upon reducing the maximum thickness of the layers to zero while allowing $N \to \infty$.

We now use the product rule following the discussion in Eqs. (4.4-7) through (4.4-15) to obtain a first-order expression for the characteristic matrix spanning a stack of layers, which will be given in terms of the reference matrix plus a first-order correction matrix. It can be shown using the Wronskian property for Airy functions, $\mathrm{AiBi}' - \mathrm{Ai}'\mathrm{Bi} = 1/\pi$, that for the reference matrix we have

$$\overline{\boldsymbol{M}}_{m,l} = \prod_{j=l+1}^{m}\overline{\boldsymbol{M}}_{j,j-1} = \begin{bmatrix}F_{m,l} & \bar{f}_{m,l}\\ \overline{G}_{m,l} & g_{m,l}\end{bmatrix} \tag{H-7}$$

From Eq. (4.4-7), it follows that a first-order expression for $\boldsymbol{M}_{N,0}$, which spans the entire stack of Airy layers, is given by

$$\begin{aligned} \boldsymbol{M}_{N,0} &= \prod_{j=1}^{N}\left(\overline{\boldsymbol{M}}_{j,j-1} + \delta\boldsymbol{M}_j\right) \\ &\doteq \overline{\boldsymbol{M}}_{N,0} + \sum_{j=1}^{N} \overline{\boldsymbol{M}}_{N,j}\delta\boldsymbol{M}_j\overline{\boldsymbol{M}}_{j,0} \end{aligned} \tag{H-8}$$

where $N$ is the number of layers in the stack. Upon carrying out the matrix multiplication in Eq. (H-8) and using the defining differential equation for Airy functions ($d^2w/dx^2 = xw$), it can be shown that the $j$th product in its limiting form is given by

$$\overline{\boldsymbol{M}}_{N,j}\delta\boldsymbol{M}_j\overline{\boldsymbol{M}}_{j,0} \to$$

$$\frac{-\pi^2}{2}\frac{d^2}{d\hat{y}_{Aj}^2}\begin{bmatrix} \begin{vmatrix}\mathrm{Ai}_N & \mathrm{Bi}_N \\ \mathrm{Ai}_j & \mathrm{Bi}_j\end{vmatrix} \cdot \begin{vmatrix}\mathrm{Ai}_j & \mathrm{Bi}_j \\ \mathrm{Ai}'_0 & \mathrm{Bi}'_0\end{vmatrix} & \dfrac{i}{\bar{\gamma}_A}\begin{vmatrix}\mathrm{Ai}_N & \mathrm{Bi}_N \\ \mathrm{Ai}_j & \mathrm{Bi}_j\end{vmatrix} \cdot \begin{vmatrix}\mathrm{Ai}_j & \mathrm{Bi}_j \\ \mathrm{Ai}'_0 & \mathrm{Bi}'_0\end{vmatrix} \\ i\bar{\gamma}_A\begin{vmatrix}\mathrm{Ai}'_N & \mathrm{Bi}'_N \\ \mathrm{Ai}_j & \mathrm{Bi}_j\end{vmatrix} \cdot \begin{vmatrix}\mathrm{Ai}_j & \mathrm{Bi}_j \\ \mathrm{Ai}'_0 & \mathrm{B}i'_0\end{vmatrix} & -\begin{vmatrix}\mathrm{Ai}'_N & \mathrm{Bi}'_N \\ \mathrm{Ai}_j & \mathrm{Bi}_j\end{vmatrix} \cdot \begin{vmatrix}\mathrm{Ai}_j & \mathrm{Bi}_j \\ \mathrm{Ai}_0 & \mathrm{Bi}_0\end{vmatrix} \end{bmatrix} \frac{\delta\gamma_{Aj}}{\bar{\gamma}_A} d\hat{y}_{Aj} \tag{H-9}$$

In shorthand notation, Eq. (H-9) can be written as

$$\overline{\boldsymbol{M}}_{N,j}\delta\boldsymbol{M}_j\overline{\boldsymbol{M}}_{j,0} \to i\frac{d^2}{d\hat{y}_{Aj}^2}\begin{bmatrix} \bar{f}_{Nj}F_{j0} & \bar{f}_{Nj}\bar{f}_{j0} \\ g_{Nj}F_{j0} & g_{Nj}\bar{f}_{j0} \end{bmatrix}\delta\gamma_{Aj}d\hat{y}_{Aj} \tag{H-10}$$

It follows that the limiting form for a first-order expression for the characteristic matrix is given by

$$\begin{aligned} &\boldsymbol{M}\left[\hat{y}_{AN}, \hat{y}_{A0}\right] \to \overline{\boldsymbol{M}}\left[\hat{y}_{AN}, \hat{y}_{A0}\right] + \\ &i\int_{\hat{y}_{A0}}^{\hat{y}_{AN}} \left(\gamma_A(z) - \bar{\gamma}_A\right)\frac{d^2}{dz^2}\begin{bmatrix} \bar{f}\left(\hat{y}_{AN}, z\right)F\left(z, \hat{y}_{A0}\right) & \bar{f}\left(\hat{y}_{AN}, z\right)\bar{f}\left(z, \hat{y}_{A0}\right) \\ g\left(\hat{y}_{AN}, z\right)F\left(z, \hat{y}_{A0}\right) & g\left(\hat{y}_{AN}, z\right)\bar{f}\left(z, \hat{y}_{A0}\right) \end{bmatrix} dz \end{aligned} \tag{H-11}$$

Here $\gamma_A = \left(2n(x)n'_A(x)/k\right)^{1/3} \neq \gamma$, so we have to be clear about the relationship between $\hat{y}_A$ and $x$ in the integration in Eq. (H-11).

For negative values of $\hat{y} <\sim -2$, we can use the asymptotic forms for the Airy functions in the characteristic matrix. Thus, $\overline{\boldsymbol{M}}\left[\hat{y}_{AN}, \hat{y}_{A0}\right]$ becomes

$$\overline{\boldsymbol{M}}\left[\hat{y}_{AN},\hat{y}_{A0}\right]\rightarrow\begin{bmatrix}\left(\dfrac{\hat{y}_{A0}}{\hat{y}_{AN}}\right)^{1/4}\cos\left(\mathscr{A}_A\right)_{N,0} & \dfrac{i\bar{\gamma}_A^{-1}}{\left(\hat{y}_{AN}\hat{y}_{A0}\right)^{1/4}}\sin\left(\mathscr{A}_A\right)_{N,0}\\ i\bar{\gamma}_A\left(\hat{y}_{AN}\hat{y}_{A0}\right)^{1/4}\sin\left(\mathscr{A}_A\right)_{N,0} & \left(\dfrac{\hat{y}_{AN}}{\hat{y}_{A0}}\right)^{1/4}\cos\left(\mathscr{A}_A\right)_{N,0}\end{bmatrix}\qquad\text{(H-12)}$$

where $\mathscr{A}_A=\mathscr{A}_A\left(\hat{y}_A,\hat{y}_{A0}\right)$ is given by

$$\left.\begin{aligned}\mathscr{A}_A\left(\hat{y}_A,\hat{y}_{A0}\right)&=\mathrm{X}\left(\hat{y}_A\right)-\mathrm{X}\left(\hat{y}_{A0}\right)\\ \mathrm{X}\left(\hat{y}_A\right)&=\frac{2}{3}\left(-\hat{y}_A\right)^{3/2}+\frac{\pi}{4}\\ \mathscr{A}_A&=k\int_{x_0}^{x}\gamma_A\sqrt{-\hat{y}_A}\,dx'=k\int_{x_0}^{x}\gamma\sqrt{-\hat{y}}\,dx'\end{aligned}\right\}\qquad\text{(H-13)}$$

The last relationship follows from Eq. (4.7-7b). From Eq. (4.5-7), it follows that $\mathscr{A}_A$ is still the phase accumulation by the wave along the x-axis, either for a wave traveling through the Airy layers or through the actual medium being modeled with Airy layers. The correction matrix in Eq. (H-11) becomes

$$\int_{\hat{y}_0}^{\hat{y}_N}\overline{\boldsymbol{M}}\left[\hat{y}_{AN},\xi\right]\left(\frac{d\boldsymbol{M}[\xi,\xi]}{d\xi}\right)\overline{\boldsymbol{M}}\left[\xi,\hat{y}_{A0}\right]d\xi\rightarrow$$

$$\int_{\hat{y}_{A0}}^{\hat{y}_{AN}}\frac{\gamma_A-\bar{\gamma}_A}{\bar{\gamma}_A}\sqrt{-\xi}\begin{bmatrix}\left(\dfrac{\hat{y}_{A0}}{\hat{y}_{AN}}\right)^{1/4}\sin\mathscr{B}_A & \dfrac{i\bar{\gamma}_A^{-1}}{\left(\hat{y}_{AN}\hat{y}_{A0}\right)^{1/4}}\cos\mathscr{B}_A\\ -i\bar{\gamma}_A\left(\hat{y}_{AN}\hat{y}_{A0}\right)^{1/4}\cos\mathscr{B}_A & -\left(\dfrac{\hat{y}_{AN}}{\hat{y}_{A0}}\right)^{1/4}\sin\mathscr{B}_A\end{bmatrix}d\xi\qquad\text{(H-14)}$$

where $\mathscr{B}_A\left(\hat{y}_{AN},\hat{y}_{A0},\xi\right)$ is given by

$$\mathscr{B}_A=\mathscr{A}_A\left(\hat{y}_{AN},\hat{y}_{A0}\right)-2\mathscr{A}_A\left(\xi,\hat{y}_{A0}\right)\qquad\text{(H-15)}$$

We now set

$$2\bar{\gamma}_A=\gamma_A\big|_{\hat{y}_{A0}}+\gamma_A\big|_{\hat{y}_{AN}}\qquad\text{(H-16)}$$

and we define

$$\left.\begin{aligned} \varpi_A &= \gamma_A \sqrt{-\hat{y}_A} \\ \bar{\varpi}_A &= \bar{\gamma}_A \left(\hat{y}_{A0}\hat{y}_{AN}\right)^{1/4} \end{aligned}\right\} \tag{H-17}$$

Thus, $k\varpi_A$ is the rate of phase accumulation along the x-axis by a wave traveling through a series of Airy layers; $k\bar{\varpi}_A$ is the geometric average of the rate of phase accumulation.

From Eqs. (H-13) and (H-15), it follows that $2(-\xi)^{1/2}$ is the derivative of $\mathscr{B}_A$ with respect to the dummy variable $\xi$; thus, one can integrate Eq. (H-14) by parts:

$$\boldsymbol{M}\left[\hat{y}_{AN}, \hat{y}_{A0}\right] \doteq \begin{bmatrix} \dfrac{\varpi_A|_{\hat{y}_{A0}}}{\bar{\varpi}_A}\cos\left(\mathscr{H}_A\right)_{N,0} + I_1 & \dfrac{i}{\bar{\varpi}_A}\left(\sin\left(\mathscr{H}_A\right)_{N,0} - I_2\right) \\ -i\bar{\varpi}_A\left(\sin\left(\mathscr{H}_A\right)_{N,0} + I_2\right) & \dfrac{\varpi_A|_{\hat{y}_{AN}}}{\bar{\varpi}_A}\cos\left(\mathscr{H}_A\right)_{N,0} - I_1 \end{bmatrix} \tag{H-18}$$

Here the integrals $I_1$ and $I_2$ are given by

$$\left.\begin{aligned} I_1 &= \frac{1}{2\bar{\gamma}_A}\int_{x_0}^{x_N} \frac{d\gamma_A}{dx}\cos\mathscr{B}_A\, dx \\ I_2 &= \frac{1}{2\bar{\gamma}_A}\int_{x_0}^{x_N} \frac{d\gamma_A}{dx}\sin\mathscr{B}_A\, dx \end{aligned}\right\} \tag{H-19}$$

These forms in Eqs. (H-18) and (H-19) are similar to those that we obtained for Cartesian stratification in Eqs. (4.4-16) and (4.4-17) for a constant intra-layer index of refraction. There, $k\varpi$ is the rate of phase accumulation of the wave along the x-axis; here $\varpi_A = k\gamma_A(x)\left(-\hat{y}_A(x)\right)^{1/2}$ also is the rate of phase accumulation along the x-axis, but for a wave traveling through a series of Airy layers. From Eq. (4.7-7b), it follows that $\varpi_A = k\gamma(x)\left(-\hat{y}(x)\right)^{1/2} = \varpi$, which may be considered as a necessary condition that the refractivity profiles in the Airy layers must satisfy to obtain a limiting form for the characteristic matrix spanning the layers that matches the exact form.